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AUTHOR(S):

Hamahata, Yoshinori

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Transformations of a series in function fields

Yoshinori Hamahata
Department of Applied Mathematics
Okayama University of Science

1. INTRODUCTION

Let

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) \quad (\text{Im}(z) > 0)$$

be the Dedekind η -function. Dedekind [2] described the transformation of $\log \eta(z)$ under the substitution $z' = (az + b)/(cz + d)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. More precisely, he proved that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $a \neq 0, c > 0$,

$$(1.1) \quad \log \eta \left(\frac{az + b}{cz + d} \right) = \log \eta(z) + \frac{1}{2} \log \left(\frac{cz + d}{i} \right) + \frac{\pi i(a + d)}{12c} - \pi i D(a, c),$$

where $D(a, c)$ is the Dedekind sum defined by

$$(1.2) \quad D(a, c) = \frac{1}{4c} \sum_{k=1}^{c-1} \cot \left(\frac{\pi ak}{c} \right) \cot \left(\frac{\pi k}{c} \right)$$

for coprime integers a and $c > 0$. We can use (1.1) to prove the so-called reciprocity law given by

$$(1.3) \quad D(a, c) + D(c, a) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac} \right)$$

for coprime positive integers a, c . Details of the proofs of (1.1) and (1.3) can be found in the book [6].

An analogy exists between number fields and function fields. For example, $A := \mathbb{F}_q[T]$, $K := \mathbb{F}_q(T)$, and $K_{\infty} := \mathbb{F}_q((1/T))$ are analogous to \mathbb{Z} , \mathbb{Q} , and \mathbb{R} , respectively. In [1] and [5], we introduced a function field analog $s(a, c)$ (see Section 2) of $D(a, c)$, and established its reciprocity law. In this report, we use the Dedekind sum $s(a, c)$ in function fields to describe the transformation of a certain series under the substitution $z' = (az + b)/(cz + d)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$. As an application, we prove the reciprocity law for $s(a, c)$.

2. REVIEW OF THE DEDEKIND SUM

Let $A = \mathbb{F}_q[T]$ and $K = \mathbb{F}_q(T)$. Let $K_{\infty} = \mathbb{F}_q((1/T))$ be the completion of K at $\infty = (1/T)$, and let C_{∞} be the completion of an algebraic closure of K_{∞} .

2.1. The Carlitz exponential function. Let $D_0 = 1$, $D_n = [n][n-1]^q \cdots [1]^{q^{n-1}}$ for $n > 0$ and $[n] = T^{q^n} - T$. Let $e(z)$ be the Carlitz exponential function defined by

$$e(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{D_n},$$

which is entire over C_{∞} . By definition, it holds that $de(z)/dz = e'(z) = 1$. The map $e : C_{\infty} \rightarrow C_{\infty}$ is \mathbb{F}_q -linear and surjective. The kernel $L := \text{Ker}(e)$ is a free A -module of rank one. It is easy to see that $e(z)$ is L -periodic: $e(z+l) = e(z)$ for $l \in L$. Let $\bar{\pi}$ denote a generator of L . The function $e(z)$ can be written as

$$e(z) = z \prod_{0 \neq l \in L} \left(1 - \frac{z}{l}\right).$$

From this, we have

$$\frac{1}{e(z)} = \frac{e'(z)}{e(z)} = \sum_{l \in L} \frac{1}{l+z}.$$

The reader is referred to Goss [4] for additional details of $e(z)$.

2.2. The Dedekind sum. Let a, c be the coprime elements of $A \setminus \{0\}$. The (inhomogeneous) Dedekind sum $s(a, c)$ is defined as

$$s(a, c) = \frac{1}{c} \sum_{0 \neq \mu \in A/cA} e\left(\frac{\bar{\pi}a\mu}{c}\right)^{-1} e\left(\frac{\bar{\pi}\mu}{c}\right)^{-1}.$$

When c is a unit of A , $s(a, c)$ is defined to be zero. This is an analog of the classical Dedekind sum $D(a, c)$ defined in (1.2). For any $\epsilon \in \mathbb{F}_q \setminus \{0\}$,

$$(2.1) \quad s(\epsilon a, c) = \epsilon^{-1} s(a, c).$$

By replacing μ with $\epsilon\mu$ ($\epsilon \in \mathbb{F}_q \setminus \{0\}$) in the definition of $s(a, c)$, we see that $s(a, c) = 0$ if $q > 3$.

3. A SERIES RELATED TO $s(a, c)$

Let $\Omega = C_{\infty} \setminus K_{\infty}$ be the Drinfeld upper half-plane, which is an analog of the classical upper half-plane $H := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. The group $GL_2(A)$ acts on Ω by fractional linear transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az + b)/(cz + d)$. Let

$$\xi(z) = \sum_{0 \neq a \in A} \frac{1}{ae(\bar{\pi}az)},$$

which is convergent for $z \in \Omega$. This can be written as

$$\xi(z) = \begin{cases} 0 & \text{if } q > 3, \\ -\sum_{a \in A_+} 1/ae(\bar{\pi}az) & \text{if } q = 3, \\ \sum_{a \in A_+} 1/ae(\bar{\pi}az) & \text{if } q = 2, \end{cases}$$

where A_+ is the set of monic elements in A .

In the classical case, it is known that for $\gamma \in SL_2(\mathbb{Z})$ and $z \in H$,

$$\log \eta(\gamma z) - \log \eta(z) = -2\pi i \int_z^{\gamma z} G_2(\tau) d\tau,$$

where $G_2(\tau) = -\frac{1}{24} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n e^{2\pi i m n \tau}$ is the Eisenstein series of weight 2. We have a similar result for $\xi(z)$. Since $de(z)/dz = 1$, $d\xi(z)/dz = \pi \sum_{a \in A_+} e(\pi a z)^{-2}$. As is well known, $g(z) := 1 - (T^q - T) \sum_{a \in A_+} e(\pi a z)^{1-q}$ is a weight $q-1$ modular form for $GL_2(A)$ (see [3], (9.2)). Therefore, we see that for $\gamma \in GL_2(A)$ and $z \in \Omega$,

$$\xi(\gamma z) - \xi(z) = \begin{cases} -\pi \int_z^{\gamma z} \frac{g(\tau)-1}{T^3-T} d\tau & \text{if } q = 3, \\ -\pi \int_z^{\gamma z} \left(\frac{g(\tau)-1}{T^2-T} \right)^2 d\tau & \text{if } q = 2. \end{cases}$$

4. TRANSFORMATION FORMULA FOR $\xi(z)$

We provide the results for the transformations for $\xi(z)$.

Proposition 1. (1) For $\epsilon \in \mathbb{F}_q \setminus \{0\}$, $\xi(\epsilon z) = \epsilon^{-1} \xi(z)$.

(2) For $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(A)$, $\xi(\gamma z) = \det \gamma \xi(z)$.

Theorem 2. We have

$$\xi(-1/z) = \xi(z) - \frac{\alpha(2)}{\pi} \left(z + \frac{1}{z} \right) - \frac{\alpha(1)^2}{\pi},$$

where $\alpha(n) = \sum_{0 \neq a \in A} a^{-n}$.

We require the following lemma to prove Theorem 4.

Lemma 3. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $GL_2(A)$ with $c \neq 0$. Then,

$$\gamma z = \frac{a}{c} - \frac{\det \gamma}{c(cz + d)}.$$

Theorem 4. For $\gamma = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \in GL_2(A)$,

$$(4.1) \quad \xi(\gamma z) = \det \gamma \left[\xi(z) - \frac{\alpha(2)}{\pi c} \left(cz + d + \frac{1}{cz + d} \right) \right] + \frac{\alpha(1)^2}{\pi}.$$

The following is the main result.

Theorem 5. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$. If $a \neq 0$ and $c \neq 0$, then

$$(4.2) \quad \xi(\gamma z) = \det \gamma \left[\xi(z) - \frac{\alpha(2)}{\pi c} \left(cz + d + \frac{1}{cz + d} \right) \right] + \frac{\alpha(1)^2}{\pi c} + \pi s(a, c).$$

5. OUTLINE OF THE PROOF OF THEOREM 5

5.1. Case $\gamma \in SL_2(A)$. Let

$$R_1 = \sum_{\substack{0 \neq f \in A \\ f \equiv 0 \pmod{c}}} \frac{1}{fe(\pi f \gamma z)} - \sum_{\substack{0 \neq f \in A \\ f \equiv 0 \pmod{c}}} \frac{1}{fe(\pi f z)},$$

$$R_2 = \sum_{\substack{0 \neq f \in A \\ f \not\equiv 0 \pmod{c}}} \frac{1}{fe(\pi f \gamma z)} - \sum_{\substack{0 \neq f \in A \\ f \not\equiv 0 \pmod{c}}} \frac{1}{fe(\pi f z)}.$$

Then, we have $\xi(\gamma z) - \xi(z) = R_1 + R_2$, for which we compute R_1 and R_2 , separately.

We first compute R_1 . By Lemma 3, $1/e(\pi f \gamma z) = 1/e(-\pi f/c(cz + d))$. Hence,

$$R_1 = \frac{1}{\pi} \sum_{\substack{0 \neq f \in A \\ f \equiv 0 \pmod{c}}} \sum_{g \in A} \frac{c(cz + d)}{f(gc(cz + d) - f)} + \frac{1}{\pi} \sum_{\substack{0 \neq f \in A \\ f \not\equiv 0 \pmod{c}}} \sum_{g \in A} \frac{1}{f(g - fz + h)}.$$

Setting $f' = f/c$, R_1 becomes

$$\frac{1}{\pi c} \sum_{0 \neq f' \in A} \sum_{g \in A} \frac{cz + d}{f'(g(cz + d) - f')} + \frac{1}{\pi c} \sum_{0 \neq f' \in A} \sum_{g \in A} \frac{1}{f'(g - f'cz + h)}.$$

We set $h = -f'd$ and then divide R_1 into the part $g = 0$ and the part $g \neq 0$. Then,

$$R_1 = -\frac{\alpha(2)}{\pi c} \left(cz + d + \frac{1}{cz + d} \right) + \frac{1}{\pi c} \sum_{0 \neq f' \in A} \sum_{0 \neq g \in A} \frac{cz + d}{f'(g(cz + d) - f')} + \frac{1}{\pi c} \sum_{0 \neq f' \in A} \sum_{0 \neq g \in A} \frac{1}{f'(g - f'(cz + d))}.$$

In the last of the two double summation terms above, we interchange f' and g . Then, R_1 becomes

$$-\frac{\alpha(2)}{\pi c} \left(cz + d + \frac{1}{cz + d} \right) + \frac{\alpha(1)^2}{\pi c}.$$

We next compute the two sums in R_2 . As for the first sum, we have

$$R_3 := \sum_{\substack{f \in A \\ f \not\equiv 0 \pmod{c}}} \frac{1}{fe(\pi f \gamma z)} = \sum_{0 \neq \mu \in A/cA} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \frac{1}{fe(\pi f \gamma z)}.$$

When $f \equiv \mu \pmod{c}$, f can be written as $f = \mu + ch$ for a certain $h \in A$. Hence, by Lemma 3,

$$\frac{\pi}{e(\pi f \gamma z)} = c \sum_{g \in A} \frac{1}{cg + a\mu - \frac{f}{cz+d}}.$$

Setting $r = cg + a\mu$, $\pi/e(\pi f \gamma z)$ can be written as

$$c(cz + d) \sum_{\substack{r \in A \\ r \equiv a\mu \pmod{c}}} \frac{1}{r(cz + d) - f}.$$

Thus, we have

$$R_3 = \frac{c}{\pi} \sum_{0 \neq \mu \in A} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \sum_{\substack{g \in A \\ g \equiv a\mu \pmod{c}}} \frac{cz + d}{f(g(cz + d) - f)}.$$

When $f \equiv \mu \pmod{c}$, using $k := a\mu - \mu$, $f + k \equiv a\mu \pmod{c}$. As for the second summation in R_2 , noting that a, c are coprime, we obtain

$$R_4 := \sum_{\substack{f \in A \\ f \not\equiv 0 \pmod{c}}} \frac{1}{fe(\pi f \gamma z)} = \sum_{0 \neq \mu \in A/cA} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \frac{1}{(f + k)e(\pi(f + k)z)}.$$

Now, we compute $\bar{\pi}/e(\bar{\pi}(f+k)z)$ as follows.

$$\frac{\bar{\pi}}{e(\bar{\pi}(f+k)z)} = -\sum_{g \in A} \frac{1}{g - (f+k)z + h},$$

where $h \in A$ is a fixed element. Because $f \equiv \mu \pmod{c}$,

$$\frac{a\mu - k - d(f+k)}{c} = \frac{-bc\mu - d(f-\mu)}{c} \in A.$$

Letting $h = (a\mu - k - d(f+k))/c$, $\bar{\pi}/e(\bar{\pi}(f+k)z)$ becomes

$$-c \sum_{\substack{r \in A \\ r \equiv a\mu \pmod{c}}} \frac{1}{r - k - (f+k)(cz+d)}.$$

Hence, we have

$$R_4 = -\frac{c}{\bar{\pi}} \sum_{0 \neq \mu \in A/cA} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \sum_{\substack{g \in A \\ g \equiv a\mu \pmod{c}}} \frac{1}{(f+k)(g-k-(f+k)(cz+d))}.$$

Noting that $f \equiv \mu \pmod{c}$ if and only if $f+k \equiv a\mu \pmod{c}$, R_4 becomes

$$-\frac{c}{\bar{\pi}} \sum_{0 \neq \mu \in A/cA} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \sum_{\substack{g \in A \\ g \equiv a\mu \pmod{c}}} \frac{1}{g(f-g(cz+d))}.$$

Using R_3 and R_4 , we obtain

$$R_2 = R_3 - R_4 = \frac{c}{\bar{\pi}} \sum_{0 \neq \mu \in A/cA} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \sum_{\substack{g \in A \\ g \equiv a\mu \pmod{c}}} \frac{1}{fg}.$$

As

$$\begin{aligned} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \frac{1}{f} &= \sum_{s \in A} \frac{1}{\mu + cs} = \frac{\bar{\pi}}{ce(\bar{\pi}\mu/c)}, \\ \sum_{\substack{g \in A \\ g \equiv a\mu \pmod{c}}} \frac{1}{g} &= \sum_{t \in A} \frac{1}{a\mu + ct} = \frac{\bar{\pi}}{ce(\bar{\pi}a\mu/c)}, \end{aligned}$$

we see that $R_2 = \bar{\pi}s(a, c)$. Therefore, we conclude that $R_1 + R_2$ is the right-hand side of (4.2).

5.2. Case $\gamma \in GL_2(A)$. The cases $q > 3$ and $q = 2$ are trivial. It suffices to show the case of $q = 3$ and $\det \gamma = -1$. Noting that $\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$ belongs to $SL_2(A)$, using the result obtained in Subsection 5.1, we have

$$\begin{aligned} \xi(\gamma z) &= \xi\left(\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}(-z)\right) \\ &= \xi(-z) - \frac{\alpha(2)}{\bar{\pi}c} \left(c(-z) - d + \frac{1}{c(-z) - d}\right) + \frac{\alpha(1)^2}{\bar{\pi}c} + \bar{\pi}s(a, c), \end{aligned}$$

which is the right-hand side of (4.2).

6. APPLICATION

In this section, we prove the following theorem.

Theorem 6 (Reciprocity law [1, 5]). *Let $a, c \in A \setminus \{0\}$ be coprime.*

(1) *If $q = 3$, then*

$$s(a, c) + s(c, a) = \frac{1}{T^3 - T} \left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac} \right).$$

(2) *If $q = 2$, then*

$$s(a, c) + s(c, a) = \frac{1}{T^4 + T^2} \left(\frac{a}{c} + \frac{c}{a} + \frac{1}{a} + \frac{1}{c} + \frac{1}{ac} + 1 \right).$$

We have already proved this theorem ([1, 5]), by computing the residues of a rational function. Using Theorem 5, we now provide another proof.

Proof of Theorem 6. There exist $b, d \in A$ such that $ad - bc = 1$. Then, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $SL_2(A)$. It holds that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma z = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} z$. We compute the values of the series ξ on both sides.

Using Theorem 2,

$$\xi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma z\right) = \xi(\gamma z) - \frac{\alpha(2)}{\pi} \left(\gamma z + \frac{1}{\gamma z} \right) - \frac{\alpha(1)^2}{\pi}.$$

Combining Lemma 3 with Theorem 5, this can be written as

$$(6.1) \quad \begin{aligned} & \xi(z) - \frac{\alpha(2)}{\pi c} \left(cz + d + \frac{1}{cz + d} \right) + \frac{\alpha(1)^2}{\pi c} + \pi s(a, c) \\ & - \frac{\alpha(2)}{\pi} \left(\frac{a}{c} - \frac{1}{c(cz + d)} + \frac{c}{a} + \frac{1}{a(az + b)} \right) - \frac{\alpha(1)^2}{\pi}. \end{aligned}$$

Using (2.1) and Theorem 5, $\xi\left(\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} z\right)$ becomes

$$(6.2) \quad \xi(z) - \frac{\alpha(2)}{\pi a} \left(az + b + \frac{1}{az + b} \right) + \frac{\alpha(1)^2}{\pi a} - \pi s(c, a).$$

Equating (6.1) with (6.2), we obtain

$$(6.3) \quad s(a, c) + s(c, a) = \frac{\alpha(2)}{\pi^2} \left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac} \right) + \frac{\alpha(1)^2}{\pi^2} \left(\frac{1}{a} - \frac{1}{c} + 1 \right).$$

Combining (6.3) with the following lemma enables us to complete the proof.

Lemma 7. (1) *If $q = 3$, then*

$$\alpha(2) = \frac{\pi^2}{T^3 - T}, \quad \alpha(1) = 0.$$

(2) *If $q = 2$, then*

$$\alpha(2) = \alpha(1)^2 = \frac{\pi^2}{T^4 + T^2}.$$

7. AN ANALOG OF THE SAWTOOTH FUNCTION

The sawtooth function $((x))$ is defined by

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$$

where $\{x\}$ is the fraction part of x . This function has the following Fourier expansion:

$$((x)) = -\frac{1}{2\pi i} \sum_{0 \neq n \in \mathbb{Z}} \frac{\exp(2\pi i n x)}{n}.$$

Inspired by the definition of $\xi(z)$, we define an analog of $((x))$ as follows. For a given $x \in K_\infty$, let S_x be the set of all $a \in A$ such that $e(\pi a x) \neq 0$. Then, we define

$$F(x) = \begin{cases} -\frac{1}{\pi} \sum_{a \in S_x} \frac{1}{ae(\pi a x)} & \text{if } S_x \neq \emptyset, \\ 0 & \text{if } S_x = \emptyset. \end{cases}$$

This function has the following properties.

- For $\epsilon \in \mathbb{F}_q \setminus \{0\}$, $F(\epsilon x) = \epsilon^{-1} F(x)$.
- For $b \in A$, $F(b) = 0$.
- For $b \in A$, $F(x + b) = F(x)$.

Moreover, the value $F(x)$ at $x \in K$ can be described in terms of the Dedekind sum in function fields:

Theorem 8. For coprime $a, c \in A \setminus \{0\}$, $F(a/c) = s(-a, c)$.

We note that for the function $((x))$, the result corresponding to Theorem 8 does not hold. It would be interesting to investigate $F(x)$ in the future.

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Department of Applied Mathematics
Okayama University of Science, Ridai-cho 1-1
Okayama, 700-0005, Japan
hamahata@xmath.ous.ac.jp

岡山理科大学・理学部 浜畑 芳紀